Journal of Number Theory **76**, 178–193 (1999)
Article ID jnth.1998.2364, available online at http://www.idealibrary.com on IDEAL®

Bernoulli Numbers and Polynomials via Residues

I-Chiau Huang

Institute of Mathematics, Academia Sinica, Nankang, Taipei 11529, Taiwan, Republic of China

and

Su-Yun Huang

Institute of Statistical Science, Academia Sinica,

Received August 27, 1997; revised October 1, 1998

An algebraic theory of residues is used to evaluate summations of the form

$$\sum_{\substack{i_1, ..., i_m \geqslant 0 \\ i_1 + \cdots + i_m = n}} j_1! \binom{i_1}{j_1} \cdots j_m! \binom{i_m}{j_m} \binom{n}{i_1, ..., i_m} B_{i_1} \cdots B_{i_m}$$

and the form

$$\sum_{\substack{i_1, ..., i_m \geqslant 0 \\ i_1 + \cdots + i_m = n}} \binom{n}{i_1, ..., i_m} N_1^{i_1} \cdots N_m^{i_m} B_{i_1}(\alpha_1) \cdots B_{i_m}(\alpha_m).$$

Various identities involving Bernoulli numbers and polynomials are derived.

Key Words: Bernoulli number; Bernoulli polynomial; residue.

1. INTRODUCTION

The purpose of this paper is to introduce an algebraic theory of residues to the study of elementary number theory. Such theory of residues was

0022-314X/99 $30.00

used by the first author to concretely realize Grothendieck duality theory [3–5]. Roughly speaking, our residue map takes coefficients of power series in an algebraic context which resembles some properties from analytic residues defined by integration. However our algebraic residues have the advantage of satisfying the vanishing law, which makes life easier.

In this paper, we concentrate on identities involving Bernoulli numbers and polynomials. Among many systematic approaches to these identities, we present a new one by establishing a framework in which Bernoulli numbers and polynomials can be represented as residues and in which algebraic operations make sense. This algebraic method was initiated in [6] and is analogous to the method of generating functions, see also [7] and [8]. However, as indicated in [6], the operation of "*taking the coefficient*" is more natural in our set-up with the presence of differentials. It is plausible that our method can also be applied to other classical numbers such as Euler numbers and Genocchi numbers. These investigations are left to the interested reader.

One of the main results of this paper is to illustrate that many identities involving Bernoulli numbers are in fact coming from the summations of the form

$$\sum_{\substack{i_1, \ldots, i_m \geqslant 0 \\ i_1 + \cdots + i_m = n}} j_1! \binom{i_1}{j_1} \cdots j_m! \binom{i_m}{j_m} \binom{n}{i_1, \ldots, i_m} B_{i_1} \cdots B_{i_m},$$

which we call *a complete sum of products of Bernoulli numbers or a complete sum* for short. We show that a complete sum of the above form equals an integer combination of Bernoulli numbers of higher orders. By lowering the orders of the Bernoulli numbers, we obtain the required identities.

Another main result of this paper is to evaluate the sum

$$\sum_{\substack{i_1, \ldots, i_m \geqslant 0 \\ i_1 + \cdots + i_m = n}} \binom{n}{i_1, \ldots, i_m} N_1^{i_1} \cdots N_m^{i_m} B_{i_1}(\alpha_1) \cdots, B_{i_m}(\alpha_m),$$

also called a complete sum, for arbitrary positive integers $N_1, \ldots, N_m$ and elements $\alpha_1, \ldots, \alpha_m$ in the underlying field. We show that such complete sum equals a sum of some values of Bernoulli polynomials. With the help of the well-known multiplication theorem, we may simplify the latter sum and obtain various identities.

In this paper, proofs of several classical theorems are included to demonstrate our method. Some results are not formulated in the most generality to keep the formulae elegant.

2. POWER SERIES RINGS AND RESIDUES

Let κ be a field of characteristic zero. For f in the maximal ideal $\kappa[[T]]\,T$ of the formal power series ring $\kappa[[T]]$, e^f is defined by

$$e^f := 1 + \frac{f}{1!} + \frac{f^2}{2!} + \frac{f^3}{3!} + \cdots.$$

Let d/dT be the operator on $\kappa[[T]]$ defined by

$$\frac{d}{dT}(a_0 + a_1 T + a_2 T^2 + \cdots) = a_1 + 2a_2 T + 3a_3 T^2 + \cdots,$$

where $a_i \in \kappa$. It is easy to see that

$$\frac{d}{dT}(e^T) = e^T$$

and

$$\frac{d}{dT}(fg) = g\frac{d}{dT}(f) + f\frac{d}{dT}(g)$$

for any $f, g \in \kappa[[T]]$. Let α be a non-zero element in κ. Although $e^{\alpha T} - 1$ is not invertible in $\kappa[[T]]$, the equation

$$(e^{\alpha T} - 1)\,X = T$$

has a unique solution, which we denote by $T/(e^{\alpha T} - 1)$. For any $\beta \in \kappa$, we write $\beta(T/(e^{\alpha T} - 1))$ as $\beta T/(e^{\alpha T} - 1)$. Indeed

$$\frac{\beta T}{e^{\alpha T} - 1} = \frac{\beta}{\alpha}\left(1 + \frac{\alpha T}{2!} + \frac{\alpha^2 T^2}{3!} + \frac{\alpha^3 T^3}{4!} + \cdots\right)^{-1}.$$

We are mainly interested in the subring of $\mathbb{Q}[[T]]$ generated by $\mathbb{Q}$, T and $T/(e^T - 1)$. The next proposition implies that $T(d/dT)$ is an operator on this subring.

PROPOSITION 1 [9, p. 62].

$$T\frac{d}{dT}\left(\frac{T}{e^T - 1}\right)^n = n\left(\frac{T}{e^T - 1}\right)^n - nT\left(\frac{T}{e^T - 1}\right)^n - n\left(\frac{T}{e^T - 1}\right)^{n+1}.$$

Proof.

$$
\begin{aligned}
n\left(\frac{T}{e^T-1}\right)^n (e^T-1)^n
&= nT^n \\
&= T\frac{d}{dT}(T^n) \\
&= T\frac{d}{dT}\left(\left(\frac{T}{e^T-1}\right)^n (e^T-1)^n\right) \\
&= nT\left(\frac{T}{e^T-1}\right)^n (e^T-1)^{n-1}\, e^T + T(e^T-1)^n \frac{d}{dT}\left(\frac{T}{e^T-1}\right)^n \\
&= nT\left(\frac{T}{e^T-1}\right)^n (e^T-1)^n + n\left(\frac{T}{e^T-1}\right)^{n+1} (e^T-1)^n \\
&\quad + T(e^T-1)^n \frac{d}{dT}\left(\frac{T}{e^T-1}\right)^n.
\end{aligned}
$$

As e^T-1 is a non-zero-divisor,

$$
n\left(\frac{T}{e^T-1}\right)^n = nT\left(\frac{T}{e^T-1}\right)^n + n\left(\frac{T}{e^T-1}\right)^{n+1} + T\frac{d}{dT}\left(\frac{T}{e^T-1}\right)^n.
$$

The result follows. ∎

In the remaining of this section, we recall some properties of generalized fractions and the residue map. The reader is referred to [6] and [4] for more details.

Let $(\widetilde{\Omega}_{\kappa[[T]]/\kappa}, d)$ be the universal separated differential module of $\kappa[[T]]$ over κ. The $\kappa[[T]]$-module $\widetilde{\Omega}_{\kappa[[T]]/\kappa}$ is freely generated by dT. Any element in the first local cohomology module $\mathrm{H}^1_{\kappa[[T]]\,T}(\widetilde{\Omega}_{\kappa[[T]]/\kappa})$ of $\widetilde{\Omega}_{\kappa[[T]]/\kappa}$ supported at the maximal ideal $\kappa[[T]]\,T$ of $\kappa[[T]]$ can be described by a generalized fraction

$$
\begin{bmatrix} f\,dT \\ T^n \end{bmatrix}
$$

for some $f\in\kappa[[T]]$ and $n>0$. Generalized fractions enjoy the following properties.

Property 1 (Linearity Law). For $f_1, f_2 \in \kappa[[T]]$, $\alpha_1, \alpha_2 \in \kappa$, and $n > 0$,

$$\begin{bmatrix} (\alpha_1 f_1 + \alpha_2 f_2)\, dT \\ T^n \end{bmatrix} = \alpha_1 \begin{bmatrix} f_1\, dT \\ T^n \end{bmatrix} + \alpha_2 \begin{bmatrix} f_2\, dT \\ T^n \end{bmatrix}.$$

The linearity law still holds if α_1 and α_2 are replaced by power series. However the above form is enough for our purpose.

Property 2 (Vanishing Law). For $f \in \kappa[[T]]$ and $n > 0$,

$$\begin{bmatrix} f\, dT \\ T^n \end{bmatrix} = 0$$

if and only if $f \in \kappa[[T]]\, T^n$.

Property 3 (Transformation Law). For $f \in \kappa[[T]]$ and $m, n > 0$,

$$\begin{bmatrix} f\, dT \\ T^n \end{bmatrix} = \begin{bmatrix} f\, T^m\, dT \\ T^{m+n} \end{bmatrix}.$$

Using the above three properties, it is easy to show that any element of $\mathrm{H}^1_{\kappa[[T]]\, T}(\tilde{\Omega}_{\kappa[[T]]/\kappa})$ can be written uniquely as a (finite) κ-linear combination of elements of the form

$$\begin{bmatrix} dT \\ T^n \end{bmatrix} \qquad (n > 0).$$

The residue map with respect to the variable T is the κ-linear map

$$\mathrm{res}_T\colon \mathrm{H}^1_{\kappa[[T]]\, T}(\tilde{\Omega}_{\kappa[[T]]/\kappa}) \to \kappa$$

satisfying

$$\mathrm{res}_T \begin{bmatrix} dT \\ T^n \end{bmatrix} = \begin{cases} 1, & \text{if} \quad n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Given $U \in \kappa[[T]]\, T \backslash \kappa[[T]]\, T^2$, the ring $\kappa[[T]]$ can be represented as a power series ring with U as a variable. So we may define res_U as well.

Property 4 (Independence of Variables). $\mathrm{res}_T = \mathrm{res}_U$.

For this reason, we can write res_T simply as res.

3. BERNOULLI NUMBERS

The Bernoulli numbers $B_i^{(n)}$ of order n are defined by

$$B_i^{(n)} = \operatorname{res}\left[\frac{i!\left(\dfrac{T}{e^T-1}\right)^n dT}{T^{i+1}}\right].$$

This definition can be traced at least back to Lucas [9, p. 60]. We write $B_i^{(1)}$ simply as B_i. The numbers B_i were discovered by Jacob Bernoulli in evaluating the sum:

$$1^n + 2^n + \cdots + m^n.$$

IDENTITY 1. *For $n \geqslant 1$ and $m \geqslant 2$,*

$$1^n + 2^n + \cdots + (m-1)^n = \sum_{i=1}^{n+1} \binom{n+1}{i} \frac{B_{n+1-i}}{n+1} m^i.$$

Proof. Since $i^n = \operatorname{res}[\frac{n!\, e^{iT}\, dT}{T^{n+1}}]$, we have

$$\begin{aligned}
&1^n + 2^n + \cdots + (m-1)^n \\
&\quad = \sum_{i=1}^{m-1} \operatorname{res}\left[\frac{n!\, e^{iT}\, dT}{T^{n+1}}\right] \\
&\quad = \operatorname{res}\left[\frac{n! \displaystyle\sum_{i=1}^{m-1} e^{iT}\, dT}{T^{n+1}}\right] \\
&\quad = \operatorname{res}\left[\frac{n!\, \dfrac{T}{e^T-1}\,(e^{mT}-e^T)\, dT}{T^{n+2}}\right] \\
&\quad = \operatorname{res}\left[\frac{n!\, \dfrac{T}{e^T-1}\, e^{mT}\, dT}{T^{n+2}}\right] - \operatorname{res}\left[\frac{n!\left(T + \dfrac{T}{e^T-1}\right) dT}{T^{n+2}}\right] \\
&\quad = \operatorname{res}\left[\frac{n!\, \dfrac{T}{e^T-1}\, e^{mT}\, dT}{T^{n+2}}\right] - \operatorname{res}\left[\frac{n!\, dT}{T^{n+1}}\right] - \operatorname{res}\left[\frac{n!\, \dfrac{T}{e^T-1}\, dT}{T^{n+2}}\right]
\end{aligned}$$

$$= \sum_{i=0}^{n+1} \binom{n+1}{i} \frac{B_{n+1-i}}{n+1} m^i - \frac{B_{n+1}}{n+1}$$

$$= \sum_{i=1}^{n+1} \binom{n+1}{i} \frac{B_{n+1-i}}{n+1} m^i. \quad \blacksquare$$

We are interested in identities involving Bernoulli numbers and polynomials. The main result of this section is the following theorem. Note that by convention $\binom{n}{i} = 0$ if $n < i$.

THEOREM 1 (Complete Sum for Bernoulli Numbers).

$$\sum_{\substack{i_1, \ldots, i_m \geqslant 0 \\ i_1 + \cdots + i_m = n}} j_1! \binom{i_1}{j_1} \cdots j_m! \binom{i_m}{j_m} \binom{n}{i_1, \ldots, i_m} B_{i_1} \cdots B_{i_m}$$

$$= \operatorname{res} \left[\frac{n!\, f_{j_1}\left(T, \frac{T}{e^T - 1}\right) \cdots f_{j_m}\left(T, \frac{T}{e^T - 1}\right) dT}{T^{n+1}} \right],$$

where polynomials $f_j \in \mathbb{Z}[U, V]$ are defined inductively as

$$f_0 := V$$

and, for $j > 0$,

$$f_j := U \frac{\partial f_{j-1}}{\partial U} + (V - UV - V^2) \frac{\partial f_{j-1}}{\partial V} - (j-1) f_{j-1}.$$

Proof. We use induction on j to show that

$$T^j \frac{d^j}{dT^j} \left(\frac{T}{e^T - 1} \right) = f_j \left(T, \frac{T}{e^T - 1} \right).$$

The case $j = 0$ is trivial. The case $j = 1$ is a special case of Proposition 1. Assume that $j \geqslant 1$ and

$$T^j \frac{d^j}{dT^j} \left(\frac{T}{e^T - 1} \right) = f_j \left(T, \frac{T}{e^T - 1} \right).$$

Apply the operator d/dT on both sides, we get

$$jT^{j-1}\frac{d^j}{dT^j}\left(\frac{T}{e^T-1}\right)+T^j\frac{d^{j+1}}{dT^{j+1}}\left(\frac{T}{e^T-1}\right)$$
$$=\frac{\partial f_j}{\partial U}\left(T,\frac{T}{e^T-1}\right)+\frac{\partial f_j}{\partial V}\left(T,\frac{T}{e^T-1}\right)\frac{d}{dT}\left(\frac{T}{e^T-1}\right).$$

Multiply both sides by T, then

$$T^{j+1}\frac{d^{j+1}}{dT^{j+1}}\left(\frac{T}{e^T-1}\right)=f_{j+1}\left(T,\frac{T}{e^T-1}\right).$$

In other words,

$$f_j\left(T,\frac{T}{e^T-1}\right)=\sum_{i\geqslant 0} j!\binom{i}{j}B_i\frac{T^i}{i!}.$$

The vanishing law can be used to simplify the index sets of some summations:

$$\sum_{\substack{i_1,\dots,i_m\geqslant 0\\ i_1+\cdots+i_m=n}} j_1!\binom{i_1}{j_1}\cdots j_m!\binom{i_m}{j_m}\binom{n}{i_1,\dots,i_m}B_{i_1}\cdots B_{i_m}$$
$$=\sum_{\substack{i_1,\dots,i_{m-1}\geqslant 0\\ i_1+\cdots+i_{m-1}\leqslant n}}\operatorname{res}\left[\frac{j_1!\binom{i_1}{j_1}\cdots j_{m-1}!\binom{i_{m-1}}{j_{m-1}}n!\frac{B_{i_1}}{i_1!}\cdots\frac{B_{i_{m-1}}}{i_{m-1}!}f_{j_m}\left(T,\frac{T}{e^T-1}\right)dT}{T^{n-i_1-\cdots-i_{m-1}+1}}\right]$$
$$=\sum_{i_1,\dots,i_{m-1}\geqslant 0}\operatorname{res}\left[\frac{n!\,j_1!\binom{i_1}{j_1}\frac{B_{i_1}T^{i_1}}{i_1!}\cdots j_{m-1}!\binom{i_{m-1}}{j_{m-1}}\frac{B_{i_{m-1}}T^{i_{m-1}}}{i_{m-1}!}f_{j_m}\left(T,\frac{T}{e^T-1}\right)dT}{T^{n+1}}\right]$$
$$=\operatorname{res}\left[\frac{n!\,f_{j_1}\left(T,\frac{T}{e^T-1}\right)\cdots f_{j_m}\left(T,\frac{T}{e^T-1}\right)dT}{T^{n+1}}\right].\ \blacksquare$$

Theorem of complete sum implies that a complete sum

$$\sum_{\substack{i_1,\dots,i_m\geqslant 0\\ i_1+\cdots+i_m=n}} j_1!\binom{i_1}{j_1}\cdots j_m!\binom{i_m}{j_m}\binom{n}{i_1,\dots,i_m}B_{i_1}\cdots B_{i_m}$$

is an integer combination of Bernoulli numbers of high orders. As a special case of the theorem, we obtain a new proof of the following identity.

IDENTITY 2.

$$\sum_{\substack{i_1, \ldots, i_m \geqslant 0 \\ i_1 + \cdots + i_m = n}} \binom{n}{i_1, \ldots, i_m} B_{i_1} \cdots B_{i_m} = B_n^{(m)}.$$

Proof.

$$\sum_{\substack{i_1, \ldots, i_m \geqslant 0 \\ i_1 + \cdots + i_m = n}} \binom{n}{i_1, \ldots, i_m} B_{i_1} \cdots B_{i_m} = \operatorname{res} \left[\frac{n! \left(\frac{T}{e^T - 1} \right)^m dT}{T^{n+1}} \right]$$
$$= B_n^{(m)}. \quad \blacksquare$$

Theorem of complete sum together with the following well-known result (see also Proposition 4) gives rise to many identities involving Bernoulli numbers.

PROPOSITION 2 (Lowering Orders) [9, Eq. (15)]. *For $n, i \geqslant 1$,*

$$B_i^{(n+1)} = \left(1 - \frac{i}{n}\right) B_i^{(n)} - i B_{i-1}^{(n)}.$$

The next identity was proved by H. Rademacher [11] using Eisenstein series and by M. Eie [2, Proposition 1] using zeta functions. We have a new proof.

IDENTITY 3. *For $n \geqslant 4$,*

$$\sum_{p=2}^{n-2} \frac{(2n-2)!}{(2p-2)!\,(2n-2p-2)!} \frac{B_{2p}}{2p} \frac{B_{2n-2p}}{2n-2p} = -\frac{(2n+1)(n-3)}{6n} B_{2n}.$$

Proof.

$$\sum_{p=2}^{n-2} \frac{(2n-2)!}{(2p-2)!\,(2n-2p-2)!} \frac{B_{2p}}{2p} \frac{B_{2n-2p}}{2n-2p}$$
$$= \frac{1}{2n(2n-1)} \sum_{i=0}^{2n} (i-1)(2n-i-1) \binom{2n}{i} B_i B_{2n-i} + \frac{1}{n} B_0 B_{2n}$$
$$- (2n-3)\, B_{2n-2} B_2$$

$$=\frac{1}{2n}\sum_{i=0}^{2n} i\binom{2n}{i} B_iB_{2n-i}-\frac{1}{2n}\sum_{i=0}^{2n}\binom{2n}{i} B_iB_{2n-i}$$

$$-\frac{1}{2n(2n-1)}\sum_{i=0}^{2n} i(i-1)\binom{2n}{i} B_iB_{2n-i}+\frac{1}{n}B_{2n}-\frac{2n-3}{6}B_{2n-2}.$$

By theorem of complete sum,

$$\sum_{p=2}^{n-2}\frac{(2n-2)!}{(2p-2)!\,(2n-2p-2)!}\,\frac{B_{2p}}{2p}\,\frac{B_{2n-2p}}{2n-2p}$$

$$=\frac{1}{2n}\operatorname{res}\left[\frac{(2n)!\,f_1\left(T,\dfrac{T}{e^T-1}\right)\dfrac{T}{e^T-1}\,dT}{T^{2n+1}}\right]-\frac{1}{2n}B_{2n}^{(2)}-\frac{1}{2n(2n-1)}$$

$$\times\operatorname{res}\left[\frac{(2n)!\,f_2\left(T,\dfrac{T}{e^T-1}\right)\dfrac{T}{e^T-1}\,dT}{T^{2n+1}}\right]+\frac{1}{n}B_{2n}-\frac{2n-3}{6}B_{2n-2}$$

$$=\frac{1}{2n}\left(B_{2n}^{(2)}-2nB_{2n-1}^{(2)}-B_{2n}^{(3)}\right)-\frac{1}{2n}B_{2n}^{(2)}-\frac{1}{2n(2n-1)}$$

$$\times\left(2n(2n-1)\,B_{2n-2}^{(2)}-4nB_{2n-1}^{(2)}-2B_{2n}^{(3)}+6nB_{2n-1}^{(3)}+2B_{2n}^{(4)}\right)$$

$$+\frac{1}{n}B_{2n}-\frac{2n-3}{6}B_{2n-2}.$$

Lowering orders of the above Bernoulli numbers, we get

$$\sum_{p=2}^{n-2}\frac{(2n-2)!}{(2p-2)!\,(2n-2p-2)!}\,\frac{B_{2p}}{2p}\,\frac{B_{2n-2p}}{2n-2p}$$

$$=-\frac{2n-3}{6n(2n-1)}B_{2n}^{(3)}-\frac{1}{2n-1}B_{2n-1}^{(3)}-\frac{2n-3}{2n-1}B_{2n-1}^{(2)}$$

$$-B_{2n-2}^{(2)}+\frac{1}{n}B_{2n}-\frac{2n-3}{6}B_{2n-2}$$

$$= \frac{(2n-3)(n-1)}{6n(2n-1)} B_{2n}^{(2)} - \frac{2n-3}{6(2n-1)} B_{2n-1}^{(2)} + \frac{1}{n} B_{2n} - \frac{2n-3}{6} B_{2n-2}$$

$$= -\frac{(2n+1)(n-3)}{6n} B_{2n}. \quad \blacksquare$$

4. BERNOULLI POLYNOMIALS

We work on the field $\kappa(X)$. The Bernoulli polynomials $B_i^{(n)}(X)$ of order n are defined by

$$B_i^{(n)}(X) = \operatorname{res}\left[\frac{i!\left(\dfrac{T}{e^T-1}\right)^n e^{XT}\,dT}{T^{i+1}}\right].$$

Since

$$B_i^{(n)}(X) = \sum_{j=0}^{i} \binom{i}{j} B_j^{(n)} X^{i-j},$$

$B_i^{(n)}(X)$ are indeed polynomials with rational coefficients. Of course, the value of $B_i^{(n)}(X)$ at any $\alpha \in \kappa$ equals

$$\operatorname{res}\left[\frac{i!\left(\dfrac{T}{e^T-1}\right)^n e^{\alpha T}\,dT}{T^{i+1}}\right].$$

We write $B_i^{(1)}(X)$ simple as $B_i(X)$. Some values of Bernoulli polynomials are easy to obtain, for instance $B_{2i+1}(\frac{1}{2}) = B_{2i+1}(1) = 0$, $B_{2i+1}(2) = 2i+1$ for $i > 0$. It is also easy to see that $B_i^{(n)}(0) = B_i^{(n)}$, $B_0^{(n)}(X) = 1$, $B_{2i}(1) = B_{2i}$, and $B_{2i}(2) = B_{2i} + 2i$. The detail is left to the reader.

Proposition 3 (Multiplication Theorem) [10, p. 21, Eq. (18)]. *If n and m are positive integers, then*

$$n^{1-m} B_m(n\alpha) = \sum_{i=0}^{n-1} B_m\left(\alpha + \frac{i}{n}\right)$$

for any $\alpha \in \kappa$.

Proof. Let $U=T/n$, then

$$\sum_{i=0}^{n-1} B_m\left(\alpha+\frac{i}{n}\right)=\sum_{i=0}^{n-1} \operatorname{res}\left[\frac{m!\,\dfrac{T}{e^T-1}\,e^{(\alpha+(i/n))\,T}\,dT}{T^{m+1}}\right]$$

$$=\operatorname{res}\left[\frac{m!\,\dfrac{T}{e^{T/n}-1}\,e^{\alpha T}\,dT}{T^{m+1}}\right]$$

$$=\operatorname{res}\left[\frac{m!\,n^{1-m}\,\dfrac{U}{e^U-1}\,e^{n\alpha U}\,dU}{U^{m+1}}\right]$$

$$=n^{1-m}B_m(n\alpha). \quad \blacksquare$$

As in the case of Bernoulli numbers, Bernoulli polynomials of high order can be represented by lower order ones:

PROPOSITION 4 (Lowering Orders) [10, p. 145, Eq. (81)]. *For positive integers n and i,*

$$B_i^{(n+1)}(X)=\left(1-\frac{i}{n}\right)B_i^{(n)}(X)+(X-n)\,\frac{i}{n}\,B_{i-1}^{(n)}(X).$$

Proof.

$$\frac{B_i^{(n)}(X)}{(i-1)!}$$

$$=\operatorname{res}\left[\frac{T\,\dfrac{d}{dT}\left(\left(\dfrac{T}{e^T-1}\right)^n e^{XT}\right)dT}{T^{i+1}}\right]$$

$$=\operatorname{res}\left[\frac{\left(n\left(\dfrac{T}{e^T-1}\right)^n-nT\left(\dfrac{T}{e^T-1}\right)^n-n\left(\dfrac{T}{e^T-1}\right)^{n+1}+T\left(\dfrac{T}{e^T-1}\right)^n X\right)e^{XT}\,dT}{T^{i+1}}\right]$$

$$=n\,\frac{B_i^{(n)}(X)}{i!}-n\,\frac{B_{i-1}^{(n)}(X)}{(i-1)!}-n\,\frac{B_i^{(n+1)}(X)}{i!}+X\,\frac{B_{i-1}^{(n)}(X)}{(i-1)!}.$$

Hence the required identity. $\blacksquare$

THEOREM 2 (Complete Sum for Bernoulli Polynomials). *Given elements* $\alpha_1, \ldots, \alpha_m \in \kappa$ *and positive integers* $N_1, \ldots, N_m, n$,

$$\sum_{\substack{i_1, \ldots, i_m \geqslant 0 \\ i_1 + \cdots + i_m = n}} \binom{n}{i_1, \ldots, i_m} N_1^{i_1} \cdots N_m^{i_m} B_{i_1}(\alpha_1) \cdots B_{i_m}(\alpha_m)$$
$$= N^{n-m} N_1 \cdots N_m$$
$$\sum_{0 \leqslant \ell_i < N/N_i} B_n^{(m)}\left((\ell_1 + \alpha_1) \frac{N_1}{N} + \cdots + (\ell_m + \alpha_m) \frac{N_m}{N}\right),$$

where N *is the least common multiplier of* $N_1, \ldots, N_m$.

Proof. For any $\beta \in \kappa$, non-negative integer i, and positive integer M,

$$\frac{M^i B_i(\beta)}{i!} = \operatorname{res}\left[\frac{M^i \dfrac{V}{e^V - 1} e^{\beta V}\, dV}{V^{i+1}}\right] = \operatorname{res}\left[\frac{\dfrac{MT}{e^{MT} - 1} e^{M\beta T}\, dT}{T^{i+1}}\right],$$

where $V = MT$, that is,

$$\frac{MT}{e^{MT} - 1} e^{M\beta T} = \sum_{i \geqslant 0} \frac{M^i B_i(\beta)}{i!} T^i.$$

Now we use the vanishing law to simplify the index set: Let $U = NT$. Then

$$\sum_{\substack{i_1, \ldots, i_m \geqslant 0 \\ i_1 + \cdots + i_m = n}} \binom{n}{i_1, \ldots, i_m} N_1^{i_1} \cdots N_m^{i_m} B_{i_1}(\alpha_1) \cdots B_{i_m}(\alpha_m)$$

$$= \sum_{i_1, \ldots, i_{m-1} \geqslant 0} \operatorname{res}$$
$$\left[\frac{n! \left(\dfrac{N_1^{i_1} B_{i_1}(\alpha_1)}{i_1!} T^{i_1}\right) \cdots \left(\dfrac{N_{m-1}^{i_{m-1}} B_{i_{m-1}}(\alpha_{m-1})}{i_{m-1}!} T^{i_{m-1}}\right) \dfrac{N_m T}{e^{N_m T} - 1} e^{N_m \alpha_m T}\, dT}{T^{n+1}}\right]$$

$$= \operatorname{res}\left[\frac{n! \dfrac{N_1 T}{e^{N_1 T} - 1} \cdots \dfrac{N_m T}{e^{N_m T} - 1} e^{N_1 \alpha_1 T + \cdots + N_m \alpha_m T}\, dT}{T^{n+1}}\right]$$

$$= \operatorname{res}\left[\frac{n!\, N^{n-m} N_1 \cdots N_m \dfrac{U}{e^{(N_1 U)/N} - 1} \cdots \dfrac{U}{e^{(N_m U)/N} - 1} e^{(N_1 \alpha_1 U + \cdots + N_m \alpha_m U)/N}\, dU}{U^{n+1}}\right].$$

As

$$\frac{U}{e^{(N_i U)/N}-1}=\frac{U}{e^{U}-1}\left(1+e^{(N_i U)/N}+e^{(2N_i U)/N}+\cdots+e^{(NU-N_i U)/N}\right)$$

for $1 \leqslant i \leqslant m$,

$$\sum_{\substack{i_1, \ldots, i_m \geqslant 0 \\ i_1+\cdots+i_m=n}} \binom{n}{i_1, \ldots, i_m} N_1^{i_1} \cdots N_m^{i_m} B_{i_1}(\alpha_1) \cdots B_{i_m}(\alpha_m)$$

$$=\sum_{\substack{0 \leqslant \ell_i < N/N_i \\ i=1, \ldots, m}} \operatorname{res}\left[\frac{n!\, N^{n-m} N_1 \cdots N_m \left(\frac{U}{e^{U}-1}\right)^m e^{((\ell_1+\alpha_1) N_1 U+\cdots+(\ell_m+\alpha_m) N_m U)/N}\, dU}{U^{n+1}}\right]$$

$$=N^{n-m} N_1 \cdots N_m \sum_{\substack{0 \leqslant \ell_i < N/N_i \\ i=1, \ldots, m}} B_n^{(m)}\left((\ell_1+\alpha_1)\frac{N_1}{N}+\cdots+(\ell_m+\alpha_m)\frac{N_m}{N}\right). \quad \blacksquare$$

The next identity is a special case of theorem of complete sum for Bernoulli polynomials.

IDENTITY 4. *For $\alpha_1, \ldots, \alpha_m \in \kappa$ and positive integers n, m,*

$$\sum_{\substack{i_1, \ldots, i_m \geqslant 0 \\ i_1+\cdots+i_m=n}} \binom{n}{i_1, \ldots, i_m} B_{i_1}(\alpha_1) \cdots B_{i_m}(\alpha_m)=B_n^{(m)}(\alpha_1+\cdots+\alpha_m).$$

We remark that K. Dilcher [1, Theorem 3] has evaluated the sum

$$\sum_{\substack{i_1, \ldots, i_m \geqslant 0 \\ i_1+\cdots+i_m=n}} \binom{n}{i_1, \ldots, i_m} B_{i_1}(\alpha_1) \cdots B_{i_m}(\alpha_m)$$

for $n \geqslant m$.

Theorem of complete sum together with proposition on lowering orders and multiplication theorem gives rise to more identities. To illustrate this method, we give a new proof of the following identity.

IDENTITY 5 [2, Proposition 3]. *For a positive integer $n > 1$, one has*

$$\sum_{k=1}^{n-1} \binom{2n}{2k} 4^{2k} 6^{2n-2k} B_{2k} B_{2n-2k} = (2^{2n} - 4^{2n} - 6^{2n} - 2^{2n+1} n) B_{2n} + (16n) 6^{2n-2} B_{2n-1}(\tfrac{1}{3})$$

Proof. By theorem of complete sum,

$$\begin{aligned}\sum_{k=1}^{n-1} \binom{2n}{2k} 4^{2k} 6^{2n-2k} B_{2k} B_{2n-2k} &= \sum_{i=0}^{2n} \binom{2n}{i} 4^{i} 6^{2n-i} B_i B_{2n-i} - (6^{2n} + 4^{2n}) B_{2n} \\ &= \frac{12^{2n}}{6} (B_{2n}^{(2)}(\tfrac{1}{3}) + B_{2n}^{(2)}(\tfrac{1}{2}) + B_{2n}^{(2)}(\tfrac{2}{3}) + B_{2n}^{(2)}(\tfrac{5}{6}) + B_{2n}^{(2)} \\ &\quad + B_{2n}^{(2)}(\tfrac{7}{6})) - (6^{2n} + 4^{2n}) B_{2n}.\end{aligned}$$

Lowering orders of the above Bernoulli polynomials, we get

$$\begin{aligned}\sum_{k=1}^{n-1} \binom{2n}{2k} 4^{2k} 6^{2n-2k} B_{2k} B_{2n-2k} &= \frac{12^{2n}(1-2n)}{6} \Big(B_{2n}(\tfrac{1}{3}) + B_{2n}(\tfrac{1}{2}) + B_{2n}(\tfrac{2}{3}) + B_{2n}(\tfrac{5}{6}) \\ &\quad + B_{2n} + B_{2n}(\tfrac{7}{6}) \Big) - 2n \frac{12^{2n}}{6} \Big(\tfrac{2}{3} B_{2n-1}(\tfrac{1}{3}) + \tfrac{1}{2} B_{2n-1}(\tfrac{1}{2}) \\ &\quad + \frac{1}{3} B_{2n-1}(\tfrac{2}{3}) + \frac{1}{6} B_{2n-1}(\tfrac{5}{6}) + B_{2n-1} - \frac{1}{6} B_{2n-1}(\tfrac{7}{6}) \Big) \\ &\quad - (6^{2n} + 4^{2n}) B_{2n}.\end{aligned}$$

By the multiplication theorem,

$$\begin{aligned}B_{2n}(\tfrac{1}{3}) + B_{2n}(\tfrac{1}{2}) + B_{2n}(\tfrac{2}{3}) + B_{2n}(\tfrac{5}{6}) + B_{2n}(1) + B_{2n}(\tfrac{7}{6}) &= 6^{1-2n} B_{2n}(2),\end{aligned}$$

$$\begin{aligned}B_{2n-1}(\tfrac{7}{6}) &= 3^{2-2n} B_{2n-1}(\tfrac{3}{2}) - B_{2n-1}(\tfrac{1}{2}) - B_{2n-1}(\tfrac{5}{6}) \\ &= 6^{2-2n} B_{2n-1}(2) - 3^{2-2n} B_{2n-1}(1) - B_{2n-1}(\tfrac{1}{2}) - B_{2n-1}(\tfrac{5}{6}),\end{aligned}$$

$$B_{2n-1}(\tfrac{5}{6}) = 2^{2-2n} B_{2n-1}(\tfrac{2}{3}) - B_{2n-1}(\tfrac{1}{3}),$$

$$B_{2n-1}(\tfrac{2}{3}) = 3^{2-2n} B_{2n-1} - B_{2n-1} - B_{2n-1}(\tfrac{1}{3}).$$

Using these identities together with $B_{2n-1} = B_{2n-1}(\frac{1}{2}) = B_{2n-1}(1) = 0$, $B_{2n-1}(2) = 2n-1$, $B_{2n}(1) = B_{2n}$, and $B_{2n}(2) = B_{2n} + 2n$, we get the required identity. ∎

Using our method, one can also evaluate the sum

$$\sum_{\substack{i_1, \dots, i_m \geqslant 0 \\ i_1 + \cdots + i_m = n}} j_1! \binom{i_1}{j_1} \cdots j_m! \binom{i_m}{j_m} \binom{n}{i_1, \dots, i_m} N_1^{i_1} \cdots N_m^{i_m} B_{i_1}(\alpha_1) \cdots B_{i_m}(\alpha_m).$$

However the formula will be a little bit lengthy.

REFERENCES

1. K. Dilcher, Sums of products of Bernoulli numbers, *J. Number Theory* **60**, No. 1 (1996), 23–41.
2. M. Eie, A note on Bernoulli numbers and Shintani generalized Bernoulli polynomials, *Trans. Amer. Math. Soc.* **348**, No. 3 (1996), 1117–1136.
3. I-C. Huang, An explicit construction of residual complexes, *J. Algebra*, to appear.
4. I-C. Huang, Pseudofunctors on modulus with zero-dimensional support, *Mem. Amer. Math. Soc.* **114**, No. 548 (1995), xii + 53.
5. I-C. Huang, Residue theorem via an explicit construction of traces, preprint 1996.
6. I-C. Huang, Applications of residues to combinatorial identities, *Proc. Amer. Math. Soc.* **125**, No. 4 (1997), 1011–1017.
7. I-C. Huang, Residue methods in combinatorial analysis, preprint 1998.
8. I-C. Huang, Reversion of power series by residues, *Comm. Algebra* **26**, No. 3 (1998), 803–812.
9. M. É. Lucas, Sur les développements en séries, *Bull. Soc. Math. France* **6** (1877), 57–68.
10. N. E. Nörlund, “Differenzenrechnung,” Springer-Verlag, Berlin/New York, 1924.
11. H. Rademacher, “Topics in Analytic Number Theory,” Springer-Verlag, Berlin/New York, 1973.
12. J. V. Uspensky and M. A. Heaslet, “Elementary Number Theory,” McGraw–Hill, New York, 1939.